ISRAEL JOURNAL OF MATHEMATICS 107 (1998), 229–235

ON THE CHARACTERISTIC POLYNOMIAL OF SUPERMATRICES

BY

JENŐ SZIGETI*
Institute of Mathematics, University of Miskolc
Miskolc-Egyetemváros, Hungary 3515
e-mail: matszj@gold.uni-miskolc.hu

ABSTRACT

We prove that the coefficients of the so-called right 2-characteristic polynomial of a supermatrix over a Grassmann algebra $G = G_0 \oplus G_1$ are in the even component G_0 of G. As a consequence, we obtain that the algebra of $n \times n$ supermatrices is integral of degree n^2 over G_0.

1. Introduction

In [K] Kemer developed a structure theory for the T-ideals of identities of associative algebras over a field of characteristic zero. The T-prime (or verbally prime) T-ideals play a fundamental role in the above theory and are in the mainstream of recent research in PI theory (see [AB, B1, B2, P, R]). Kemer proved that any T-prime T-ideal can be obtained as the T-ideal of identities of one of the following algebras: $M_n(K)$, $M_n(G)$ and $M_{n,u}(G)$, where $n > u \geq 1$ are arbitrary integers, K is the base field, G is an infinite dimensional Grassmann algebra over K, while $M_n(K)$ and $M_n(G)$ are algebras of $n \times n$ matrices over K and G, respectively. Our attention is now focused on the third object in the above list: $M_{n,u}(G)$ is the algebra of $n \times n$ supermatrices over $G = G_0 \oplus G_1$ with G_0 blocks of sizes $u \times u$ and $(n-u) \times (n-u)$ and with G_1 blocks of sizes $u \times (n-u)$ and $(n-u) \times u$. The standard notation for $M_{n,u}(G)$ is $M_{u,v}(G)$ with $v = n-u$, however in our notation it is more explicit that we deal with $n \times n$ matrices. One of the starting points in the investigation of the identities of $M_n(K)$ is the classical Cayley–Hamilton theorem. Recently the author introduced new

* Partially supported by OTKA of Hungary, grant no. T16432.
Received December 2, 1996 and in revised form May 20, 1997

determinants for $n \times n$ matrices over an arbitrary ring. It turned out that these determinants are extremely useful in the case when the base ring is Lie-nilpotent. Since G is Lie-nilpotent of index 2, the theory presented in [Sz] can be applied to $M_n(G)$. The so-called right (or left) 2-characteristic polynomial $p_A(x)$ of a matrix $A \in M_n(G)$ and the corresponding Cayley–Hamilton identity satisfied by A also seem to occupy a central position in the study of the various identities of $M_n(G)$. In general $p_A(x)$ is a polynomial with coefficients in G, i.e. $p_A(x) \in G[x]$ (x is a commuting indeterminate). On imposing certain assumptions on the shape of a matrix, it is natural to expect its characteristic polynomial to have some nice extra properties (e.g. the characteristic polynomial of a symplectically symmetric matrix is a full square). The main aim of the present note is to prove that the coefficients of the right 2-characteristic polynomial of a supermatrix are central. More precisely, we show that if $A \in M_{n,u}(G)$ then $p_A(x) \in G_0[x]$. A remarkable consequence of this fact (and of the 2-Lie nilpotent Cayley–Hamilton identity in [Sz]) is that the subalgebra $M_{n,u}(G)$ in $M_n(G)$ is integral of degree n^2 over G_0. This is an improvement of the bound $2n^2$ which is the best known (and probably the exact) upper bound for the degree of integrality of $M_n(G)$ over G_0 (see also in [Sz]). A further consequence is that $M_{n,u}(G)$ satisfies a "stronger" identity of algebraicity than the whole $M_n(G)$. In order to give a self-contained exposition, here we provide all the necessary prerequisites from [Sz].

2. Determinants and characteristic polynomials for $n \times n$ matrices

First we recall the definition of the preadjoint. Let $A = [a_{ij}] \in M_n(R)$ be an $n \times n$ matrix over an arbitrary ring R. For the permutations $\rho \in \mathrm{Sym}(\{1, 2, \ldots, n\})$ and $\tau \in \mathrm{Sym}(\{1, \ldots, s-1, s+1, \ldots, n\})$ we shall make use of the following product:

$$a(s, \tau, \rho) = a_{\tau(1)\rho(\tau(1))} \cdots a_{\tau(s-1)\rho(\tau(s-1))} a_{\tau(s+1)\rho(\tau(s+1))} \cdots a_{\tau(n)\rho(\tau(n))}.$$

The (two-sided) preadjoint of A is the matrix $A^* = [a^*_{rs}] \in M_n(R)$, where

$$a^*_{rs} = \sum_{\tau, \rho} \mathrm{sgn}(\rho) a(s, \tau, \rho), \quad 1 \le r, s \le n$$

and the sum is taken over all permutations $\tau \in \mathrm{Sym}(\{1, \ldots, s-1, s+1, \ldots, n\})$ and $\rho \in \mathrm{Sym}(\{1, 2, \ldots, n\})$ with $\rho(s) = r$. The right adjoint sequence $(A_k)_{k \ge 1}$ of A is defined by the following recursion: $A_1 = A^*$, and for $k \ge 1$ let

$$A_{k+1} = (AA_1 \cdots A_k)^*.$$

For an integer $m \geq 1$, the right m-determinant $\mathrm{rdet}_{(m)}(A) \in R$ of A is the $(1,1)$ entry of the product matrix $AA_1 \cdots A_m$. We note that another possible definition of $\mathrm{rdet}_{(m)}(A)$ is $\frac{1}{n}\mathrm{tr}(AA_1 \cdots A_m)$, which coincides with the $(1,1)$ entry of $AA_1 \cdots A_m$ for an m-Lie nilpotent R (see [D]). Let $R[x]$ denote the ring of polynomials of the commuting (central) indeterminate x, with coefficients in R. The right m-characteristic polynomial of A is defined as the right m-determinant of the matrix $A - Ex \in M_n(R[x])$, where $E \in M_n(R)$ is the unit matrix:

$$\begin{aligned} p_A(x) &= \lambda_0 + \lambda_1 x + \cdots + \lambda_d x^d \\ &= \mathrm{rdet}_{(m)}(A - Ex) \in R[x], \quad \lambda_0, \lambda_1, \ldots, \lambda_d \in R, \quad \lambda_d \neq 0. \end{aligned}$$

It is not hard to show that $d = n^m$ and $\lambda_d = (-1)^n[(n-1)!]^{1+n+n^2+\cdots+n^{m-1}}$, $\lambda_0 = \mathrm{rdet}_{(m)}(A)$ (see Proposition 4.1 in [Sz]).

A ring R is called m-Lie nilpotent if $[[[\cdots[[u_1, u_2], u_3], \ldots], u_m], u_{m+1}] = 0$ for all $u_1, u_2, u_3, \ldots, u_m, u_{m+1} \in R$ (here $[u,v] = uv - vu$). One of the main results in [Sz] is the following.

THEOREM 2.1: *If $p_A(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_d x^d$ is the right m-characteristic polynomial of an $n \times n$ matrix $A \in M_n(R)$ over an m-Lie nilpotent ring R, then the left substitution of A into $p_A(x)$ is zero: $(A)p_A = E\lambda_0 + A\lambda_1 + \cdots + A^d\lambda_d = 0$.*

3. Supermatrices over $\mathbf{Z}_2$-graded algebras

Let R be an algebra over a field K, then the pair (R_0, R_1) is called a $\mathbf{Z}_2$-grading of R if R_0 and R_1 are K-subspaces of R with the properties: $R_0 \oplus R_1 = R$, $R_0^2, R_1^2 \subseteq R_0$ and $R_0R_1, R_1R_0 \subseteq R_1$. Note that the condition $R_0^2 \subseteq R_0$ implies that R_0 is a subalgebra of R. The elements of $R_0 \cup R_1$ are called homogeneous, the parity of a homogeneous element is even if it is in R_0 and odd if it is in R_1 ($R_0 \cap R_1 = \{0\}$). The Grassmann algebra $G = K\langle v_1, v_2, \ldots \mid v_iv_j + v_jv_i = 0\rangle$ has a natural $\mathbf{Z}_2$-grading (G_0, G_1), where G_0 is the K-subspace of G generated by the monomials of even length and G_1 is the K-subspace of G generated by the monomials of odd length. Clearly, this (G_0, G_1) has the following additional property: $g_0g = gg_0$ and $g_1h_1 = -h_1g_1$ for all $g \in G$, $g_0 \in G_0$ and $g_1, h_1 \in G_1$. A supermatrix is an $n \times n$ matrix $A \in M_n(R)$ which can be partitioned into square and rectangular blocks as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is a $u \times u$ and A_{22} is an $(n-u) \times (n-u)$ square matrix over R_0, while A_{12} is a $u \times (n-u)$ and A_{21} is an $(n-u) \times u$ rectangular matrix over R_1. For the integers $n > u \geq 1$ and for a fixed $\mathbf{Z}_2$-grading (R_0, R_1) of R, the set of all such supermatrices is denoted by $M_{n,u}(R)$. It is easy to see that $M_{n,u}(R)$ is a subalgebra of $M_n(R)$. Our results are built on the following crucial observation.

THEOREM 3.1: *If $A \in M_{n,u}(R)$ is a supermatrix, then we have* $\mathrm{rdet}_{(m)}(A) \in R_0$ *for the right m-determinant of A.*

Proof: First we claim that $M_{n,u}(R)$ is closed under the formation of the preadjoint, i.e. that we have $A^* \in M_{n,u}(R)$ for $A \in M_{n,u}(R)$. The pair (i,j) is called the position of the entry a_{ij} in the matrix A. For $k,l \in \{1,2\}$ and $\rho \in \mathrm{Sym}(\{1,2,\ldots,n\})$ let $A_{kl}(\rho)$ denote the number of positions $(t,\rho(t))$, $1 \leq t \leq n$ falling into the block of A_{kl}, e.g. $A_{11}(\rho) = |\{(t,\rho(t)) \mid 1 \leq t \leq u, 1 \leq \rho(t) \leq u\}|$. Since ρ is a permutation, each row and each column of A contains exactly one position of the form $(t,\rho(t))$. In consequence, we get that $A_{12}(\rho) = A_{21}(\rho) = u - A_{11}(\rho)$. Thus the number of positions $(t,\rho(t))$, $1 \leq t \leq n$ falling into one of the blocks of A_{12} and A_{21} is even: $A_{12}(\rho) + A_{21}(\rho) = 2(u - A_{11}(\rho))$.

In view of $R_0^2, R_1^2 \subseteq R_0$ and $R_0R_1, R_1R_0 \subseteq R_1$, it is clear that the product $a(s,\tau,\rho)$ is homogeneous and its parity equals the parity of the number of positions $(t,\rho(t))$, $1 \leq t \leq n$, $t \neq s$ falling into one of the blocks of A_{12} and A_{21}. This number is $2(u - A_{11}(\rho)) - \Phi(s)$, where $\Phi(s) = 1$ if the position $(s,\rho(s))$ belongs to one of the blocks of A_{12} and A_{21}, while $\Phi(s) = 0$ if the position $(s,\rho(s))$ belongs to one of the blocks of A_{11} and A_{22}. Since $\Phi(s)$ and the element $a_{s\rho(s)} \in R_0 \cup R_1$ are of the same parity, we get that each of the summands in

$$a^*_{rs} = \sum_{\tau,\rho} \mathrm{sgn}(\rho) a(s,\tau,\rho)$$

is homogeneous of the same parity as a_{sr}. It follows that a^*_{rs} is also homogeneous and its parity equals to the parity of a_{sr}. To prove the claim, it is enough to note that the entries a_{sr} and a_{rs} in the supermatrix A have the same parity. Since $M_{n,u}(R)$ is closed under multiplication, our claim ensures that the matrices in the right adjoint sequence $(A_k)_{k \geq 1}$ of A are in $M_{n,u}(R)$. We also have $AA_1 \cdots A_m \in M_{n,u}(R)$ and thus the $(1,1)$ entry (as well as the trace) of $AA_1 \cdots A_m$ is in R_0. ∎

COROLLARY 3.2: *If $A \in M_{n,u}(G)$ is a supermatrix with respect to the natural $\mathbf{Z}_2$-grading (G_0, G_1) of the Grassmann algebra G, then the coefficients of the right 2-characteristic polynomial $p_A(x) = \mathrm{rdet}_{(2)}(A - Ex)$ are in G_0, i.e. $p_A(x) \in G_0[x]$.*

Proof: The polynomial algebra $G[x]$ also has a natural $\mathbf{Z}_2$-grading $(G_0[x], G_1[x])$, arising from the $\mathbf{Z}_2$-grading (G_0, G_1) of G. Since $A \in M_{n,u}(G)$, the diagonal entries in $A - Ex$ are of the form $g_0 - x$ with $g_0 \in G_0$. In view of $g_0 - x \in G_0[x]$, $G_0 \subseteq G_0[x]$ and $G_1 \subseteq G_1[x]$, we get that $A - Ex \in M_{n,u}(G[x])$ is a supermatrix with respect to the $\mathbf{Z}_2$-grading $(G_0[x], G_1[x])$. The application of Theorem 3.1 gives that $p_A(x) = \mathrm{rdet}_{(2)}(A - Ex) \in G_0[x]$, i.e. that the coefficients of $p_A(x)$ are in G_0. ∎

The commutative K-algebra G_0 can be viewed as a "diagonal" subalgebra in $M_n(G)$ and thus $M_n(G)$ becomes an algebra over G_0. In general the coefficients of the right 2-characteristic polynomial of a matrix $A \in M_n(G)$ are not in G_0 and so we cannot use Theorem 2.1 directly for proving the integrality of $M_n(G)$ over G_0. However, the trick applied on $p_A(x) = \mathrm{rdet}_{(2)}(A - Ex)$ in the last section of [Sz] gives that $M_n(G)$ is integral of degree $2n^2$ over G_0. Now we derive a stronger result for supermatrices; here an immediate application of Theorem 2.1 and Corollary 3.2 provides the bound n^2 for the degree of integrality of $M_{n,u}(G)$ over G_0.

THEOREM 3.3: *The algebra of $n \times n$ supermatrices $M_{n,u}(G)$ over the Grassmann algebra $G = K\langle v_1, v_2, \ldots \rangle$ is integral of degree n^2 over G_0 (the field K is of characteristic zero).*

Proof: The 2-Lie nilpotency of G enables us to use Theorem 2.1 and so we get that

$$E\lambda_0 + A\lambda_1 + \cdots + A^d\lambda_d = 0,$$

where $d = n^2$, $\lambda_d = [(n-1)!]^{1+n}$ and by Corollary 3.2 we have $\lambda_t \in G_0$, $0 \le t \le d$ for the coefficients of the right 2-characteristic polynomial $p_A(x) = \mathrm{rdet}_{(2)}(A - Ex)$ of the supermatrix $A \in M_{n,u}(G)$. Since $chr(K) = 0$, the leading coefficient λ_d is a nonzero element of K. On multiplying by λ_d^{-1}, we get that

$$E(\lambda_0\lambda_d^{-1}) + A(\lambda_1\lambda_d^{-1}) + \cdots + A^{d-1}(\lambda_{d-1}\lambda_d^{-1}) + A^d = 0$$

and each of the coefficients $\lambda_t\lambda_d^{-1}$, $0 \le t \le d$ is in G_0. This completes the proof. ∎

Remark 3.4: The "relative" Cayley–Hamilton equation (4.17) or (4.19) in [KT] gives that the algebra of 2×2 supermatrices $M_{2,1}(G)$ is integral of degree 3 over G_0. In general, the coefficients of the Cayley–Hamilton polynomial (4.15) are not in G_0 (see (4.18) or (4.20)) and thus cannot be used to verify the integrality of $M_{n,u}(G)$ over G_0. The so-called invariant Cayley–Hamilton polynomial (see also in [KT]) of an $A \in M_{n,u}(G)$ is in $G_0[x]$, moreover its coefficients are polynomial expressions of the supertraces $\mathrm{str}(A^k)$, $k \ge 1$. The well known embedding of $M_u(G)$ into $M_{2u,u}(G)$ (see [AB]) shows that this invariant Cayley–Hamilton identity for $M_{2u,u}(G)$ still cannot be considered as the verification of the integrality of $M_{2u,u}(G)$ over G_0. If $B \in M_{2u,u}(G)$ denotes the image of an $A \in M_u(G)$, then $A = A_0 + A_1$ with $A_0 \in M_u(G_0)$, $A_1 \in M_u(G_1)$ and

$$B = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}.$$

Now $\mathrm{str}(B^k) = 0$ for all $k \geq 0$ and for the above B both forms (6.1) and (6.10) of the invariant Cayley–Hamilton equation in [KT] have only zero coefficients (for $u = 1$ see (6.28)). A slightly modified construction, using zeros in the middle horizontal and in the middle vertical stripes, gives a similar B in $M_{n,u}(G)$ even if $n \neq 2u$.

The standard polynomial in the non-commuting indeterminates $y_1, y_2, \ldots, y_N$ is defined as

$$S_N(y_1, y_2, \ldots, y_N) = \sum_{\pi \in \mathrm{Sym}(\{1,2,\ldots,N\})} \mathrm{sgn}(\pi) y_{\pi(1)} y_{\pi(2)} \cdots y_{\pi(N)}$$

and we use it in the formulation of our final result.

COROLLARY 3.5: *$S_{n^2}([Y^{n^2}, Z], [Y^{n^2-1}, Z], \ldots, [Y^2, Z], [Y, Z]) = 0$ is a polynomial identity on the algebra $M_{n,u}(G)$ of $n \times n$ supermatrices over the (infinite dimensional) Grassmann algebra* ($\mathrm{char}(K) = 0$).

Proof: Using the multilinear and alternating properties of S_{n^2}, it follows immediately from Theorem 3.3. ∎

It would be important to prove that the degree of the integrality of $M_n(G)$ and $M_{n,u}(G)$ over G_0 is exactly $2n^2$ and n^2 (for $n \geq 3$), respectively. Clearly, it is enough to show that $2n^2$ (n^2) is the minimal k for which

$$S_k([Y^k, Z], [Y^{k-1}, Z], \ldots, [Y^2, Z], [Y, Z]) = 0$$

is an identity of $M_n(G)$ (of $M_{n,u}(G)$ for $n \geq 3$).

ACKNOWLEDGEMENT: The author thanks Mátyás Domokos for helpful comments on the manuscript.

References

[AB] B. Anderson and A. Berele, *Not all graded identities for $M_n(E)$ comes from $M_{n,n}$*, Communications in Algebra **23** (1995), 177–184.

[B1] A. Berele, *Trace identities and $\mathbf{Z}/2\mathbf{Z}$-graded invariants*, Transactions of the American Mathematical Society **309** (1988), 581–589.

[B2] A. Berele, *Supertraces and matrices over Grassmann algebras*, Advances in Mathematics **108** (1994), 91–103.

[D] M. Domokos, *Cayley–Hamilton theorem for 2×2 matrices over the Grassmann algebra*, Journal of Pure and Applied Algebra (1998), to appear.

[KT] I. Kantor and I. Trishin, *On a concept of determinant in the supercase*, Communications in Algebra **22** (1994), 3679–3739.

[K] A. R. Kemer, *Varieties of $\mathbf{Z}_2$-graded algebras*, Mathematics of the USSR-Izvestiya **25** (1985), 359–374.

[P] A. Popov, *On the identities of the matrices over the Grassmann algebra*, Journal of Algebra **168** (1994), 828–852.

[R] Yu. Razmyslov, *Identities with a trace and central polynomials in matrix superalgebras $M_{n,k}$*, Mathematicheskii Sbornik **128** (1985), 194–215.

[Sz] J. Szigeti, *New determinants and the Cayley–Hamilton theorem for matrices over Lie nilpotent rings*, Proceedings of the American Mathematical Society **125** (1997), 2245–2254.